Chapter 2

Bifurcations

In the previous chapter we discuss the fact that dissipative systems possess attractors. Generally, in a model, the vector field $\vec{F}$ depends on parameters and the behavior of the system may depend on the values of those parameters. It means that the number and/or nature of attractors may change when parameters are tuned. Such a change is called a bifurcation. The value of the parameter for which such qualitative change of the structure of the phase space happens is called the critical value of the parameter.

The bifurcation theory studies this problem for any numbers of parameters. Here we will study only the simple case of a vector field $\vec{F}$ depending only on one parameter. Moreover, we will only study the cases where a limited part of the phase space is involved in the change. Those particular kind of bifurcations are local and of codimension 1.

Near a bifurcation, the dynamical system can be reduced to a generic mathematical form by a change of variables and a reduction of its dimension to keep only the directions implied in the bifurcation. Those reduced mathematical expressions are called normal forms and each form is associated to a type of bifurcation.

2.1 Saddle-node bifurcation

This bifurcation corresponds to the apparition or anihilation of a pair of fixed points. Its normal form is:

$$\dot{x} = \mu - x^2$$

where $x \in \mathbb{R}$.

The study of the stability of the normal form gives two fixed points $\pm \sqrt{\mu}$ which can exist only when $\mu \geq 0$. As $\frac{\partial F}{\partial x} = -2x$, we obtain $\frac{\partial F}{\partial x}\big|_{+\sqrt{\mu}} = -2\sqrt{\mu} < 0$ and $\frac{\partial F}{\partial x}\big|_{-\sqrt{\mu}} = 2\sqrt{\mu} > 0$ so that the fixed point $+\sqrt{\mu}$ is stable and the other one $-\sqrt{\mu}$ is unstable.
Those fixed points are then plotted in a graph as a function of the value of the parameter. We call such graph *bifurcation diagram*. The set of stable fixed points is drawn with a solid line, the set of unstable fixed points is drawn with a dashed line.

Figure 2.1: Saddle-node bifurcation. The solid red line is the function \( x^* = +\sqrt{\mu} \) corresponding to the set of stable stationary solutions, the dashed one \( x^* = -\sqrt{\mu} \) giving the set of unstable stationary solutions.

We obtain *branches of solutions*. A point of the bifurcation diagram from which several branches emerge is a bifurcation point. In the case of Figure 2.1, the bifurcation point is 0 and the critical value of the parameter \( \mu \) is also 0.

For each normal form, an inverse bifurcation is obtained by changing the sign of the nonlinearity. Here the inverse bifurcation is:

\[
\dot{x} = \mu + x^2
\]

Figure 2.2: Inverse saddle-node bifurcation.

### 2.2 Transcritical bifurcation

This bifurcation corresponds to an exchange of stability between two fixed points. Its normal form is:

\[
\dot{x} = \mu x - x^2
\]

The fixed points are \( x^* = 0 \) and \( x^* = \mu \). \( \frac{dF}{dx} = \mu - 2x \), which gives:
2.3 Pitchfork bifurcation

2.3.1 Supercritical bifurcation

The normal form of the bifurcation has the symmetry \( x \rightarrow -x \):

\[
\dot{x} = \mu x - x^3.
\]

We can find 1 or 3 fixed points depending on the sign of \( \mu \): the fixed point \( x^* = 0 \) exists for all values of \( \mu \), while the two symmetric fixed points \( \pm \sqrt{\mu} \) exist only for \( \mu > 0 \).

The stability analysis gives:

- \( \left\{ \frac{dF}{dx} \right\}_0 = \mu \), the fixed point 0 is stable when \( \mu < 0 \) and unstable when \( \mu > 0 \),
- \( \left\{ \frac{dF}{dx} \right\}_\mu = -\mu \), the fixed point \( \mu \) is unstable when \( \mu < 0 \) and stable when \( \mu > 0 \).

\[
\left\{ \frac{dF}{dx} \right\}_{\pm \sqrt{\mu}} = -2\mu < 0, \text{ the two symmetric fixed point } \pm \sqrt{\mu} \text{ are stable when they exist.}
\]

Figure 2.3: Transcritical bifurcation.

Figure 2.4: Supercritical pitchfork bifurcation.
When the system crosses the bifurcation coming from $\mu < 0$, it has to choose one of the two stable branches of the pitchfork. This choice is called a symmetry breaking: the chosen solution has lost the symmetry $x \to -x$.

### 2.3.2 Subcritical bifurcation

In the supercritical case, the nonlinear term $-x^3$ saturates the linear divergence for $\mu > 0$, leading to two new stable solutions when $x^* = 0$ becomes unstable. In the subcritical (inverse) case, the nonlinear term is $+x^3$ is destabilizing:

$$\dot{x} = \mu x + x^3$$

Consequently, to obtain stable solutions for $\mu > 0$, one has to take into account higher order terms in the normal form. Keeping the symmetry $x \to -x$ of the system, we get:

$$\dot{x} = \mu x + x^3 - x^5$$

(2.1)

To obtain this bifurcation diagram, one has to study the fixed points of eq (2.1) and their stability.

The solutions of $\mu x + x^3 - x^5 = 0$ are either $x^* = 0$ or the solutions of the polynomial $\mu + x^2 - x^4 = 0$. Let $y = x^2$, we search the solutions of $y^2 - y - \mu = 0$ which are $y_\pm = \frac{1 \pm \sqrt{1 + 4 \mu}}{2}$ for $\mu \geq -1/4$. As $y = x^2$, we can use the $y_\pm$ only when they are positive.

- $y_+ = \frac{1 + \sqrt{1 + 4 \mu}}{2}$ is always positive in its existence domain $\mu \geq -1/4$
- $y_- = \frac{1 - \sqrt{1 + 4 \mu}}{2}$ is positive only when $1 \geq \sqrt{1 + 4 \mu}$, i.e. when $-1/4 \leq \mu \leq 0$.

To summarize:

- when $\mu < -1/4$ there is only one stationary solution, $x^* = 0$, and as $\frac{dF}{dx}\big|_0 = \mu < 0$, it is stable.
- when $-1/4 \leq \mu \leq 0$, there are five fixed points. $x^* = 0$ still exists and is still stable. The four other fixed points are given by $\pm \sqrt{y_+}$ and $\pm \sqrt{y_-}$. We will not detail the linear stability analysis but you can make the calculation and show that the two solutions $\pm \sqrt{\frac{1 - \sqrt{1 + 4 \mu}}{2}}$ are unstable while $\pm \sqrt{\frac{1 + \sqrt{1 + 4 \mu}}{2}}$ are stable.
• when $\mu \geq 0$, only 3 fixed points remain because the solutions $\pm \sqrt{y}$ do not exist anymore. The fixed point $x^* = 0$ is now unstable as $\mu > 0$, while the solutions $\pm \sqrt{1+\frac{\sqrt{1+4\mu}}{2}}$ are still stable.

![Figure 2.5: Subcritical pitchfork bifurcation.](image)

In the interval $[-1/4; 0]$, several stable solutions coexist: there is bistability between the solutions. The choice between the 0 solution and one or the other of the symmetric stable branches depends on the history of the system. An hysteresis cycle will be observed when the parameter $\mu$ is tuned one way and the other around the values $-1/4$ and 0.

## 2.4 Hopf bifurcation

This bifurcation corresponds to the emergence of a periodic solution from a stationary solution.

For all the previous bifurcations between different fixed points, a one-dimensional normal form was sufficient to describe each bifurcation. But now, as a limit cycle is a bidimensional object, one needs a bidimensional normal form to describe the bifurcation. This is taken into account by writing an equation on a complex:

$$\dot{z} = (\mu + i\gamma)z - z|z|^2$$  \hspace{1cm} (2.2)\]

with $z \in \mathbb{C}$, $\mu$ and $\gamma$ in $\mathbb{R}$ and $\gamma \neq 0$.

Fixed points are given by: $(\mu + i\gamma)z = |z|^2z$, and if $\gamma \neq 0$, the only solution is $z = 0$ (because $|z|^2 \in \mathbb{R}$).
To do the linear stability analysis, we want more conventional writing of the equations with two equations on variables in $\mathbb{R}$. Writing $z = x + iy$, we have:

$$
\begin{align*}
\dot{x} &= \mu x - \gamma y - x(x^2 + y^2) \\
\dot{y} &= \mu y + \gamma x - y(x^2 + y^2)
\end{align*}
$$

The linearization in $(0,0)$ gives

$$
L|_{(0,0)} = \begin{bmatrix} \mu & -\gamma \\ \gamma & \mu \end{bmatrix},
$$

from which we deduce $\Delta = \mu^2 + \gamma^2 > 0$ and $T = 2\mu$. Consequently, the fixed point $(0,0)$ is stable for $\mu < 0$ and unstable for $\mu > 0$. At the bifurcation, the behavior around the fixed point changes from a convergent spiral to a divergent spiral. To understand what happens to the trajectories when $\mu > 0$, we use the other decomposition of a complex number with modulus and phase: $z = re^{i\theta}$ which gives:

$$
\begin{align*}
\dot{r} &= \mu r - r^3 \\
\dot{\theta} &= \gamma
\end{align*}
$$

We recognize for the evolution of the modulus the normal form of a supercritical pitchfork bifurcation, while the phase has a linear dependence in time. We deduce from this system that the stable solution for $\mu > 0$ is a solution of fixed modulus but linearly increasing phase with time. It is a periodic solution which verifies $r = \sqrt{\mu}$ and $\theta = \gamma t + \theta_0$. The bifurcation diagram of a Hopf bifurcation is given in Figure 2.6.

![Hopf bifurcation diagram](image)

Figure 2.6: Hopf bifurcation.

The subcritical case exists: $\dot{z} = (\mu + i\gamma)z + z|z|^2$. The study is exactly the same as the supercritical case, leading to unstable limit cycles solution for $\mu < 0$. 
2.5. IMPERFECT BIFURCATION. INTRODUCTION TO CATASTROPHE THEORY

2.5 Imperfect bifurcation. Introduction to catastrophe theory

Bibliography:

• Nonlinear dynamics and chaos, S. Strogatz,
• Dynamiques complexes et morphogénèse, C. Misbah.

Coming back to the pitchfork bifurcation, you can have the intuition that the bifurcation diagram we draw was idealized and that in fact, in a real system a branch will be always chosen preferentially because of some imperfection of the system.

This problematic deals with the question of the robustness of the model to a perturbation (note that we speak of the perturbation of the model and not of a given solution of the equations).

In this part, we want to know how the pitchfork bifurcation will be modified if a new parameter $h$ is added to its normal form:

$$\dot{x} = \mu x - x^3 + h$$

Do we observe the same qualitative bifurcation or not? Does it keep its general properties?

Let’s find the fixed points of this new equation. They are given by the intersection of the curve $y = g(x)$ with $g(x) = x^3 - \mu x$ with the line $y = h$.

By studying the function $g(x)$, we obtain that $g(x)$ is strictly growing when $\mu < 0$, so that there is a unique fixed point $x^*$ for given values of the parameters $\mu$ and $h$. For
\( \mu > 0 \), the derivative of \( g(x) \) is negative in the interval \([−\sqrt{\frac{\mu}{3}}, +\sqrt{\frac{\mu}{3}}]\), so that depending on the value of \( h \), we can find either 1 or 3 fixed points (respectively blue and green cases on the right part of Figure 2.7). A straightforward calculation gives that we find 3 fixed points when \( h \in \left[−\frac{2\mu}{3}\sqrt{\frac{\mu}{3}}, +\frac{2\mu}{3}\sqrt{\frac{\mu}{3}}\right] \), while for values of \( h \) outside this interval there is only one fixed point.

In the space of the parameters \((\mu, h)\), we can then draw a **stability diagram** as shown in Figure 2.8. The hatched area is delimited by the curves \( h = \frac{2\mu}{3}\sqrt{\frac{\mu}{3}} \) and \( h = −\frac{2\mu}{3}\sqrt{\frac{\mu}{3}} \).

![Figure 2.8: Stability diagram of the imperfect pitchfork bifurcation.](image)

Their junction point in \((0, 0)\) is a singular point called a **cusp**.

To find the stability of the fixed points, we can use the fact that if \( F(x) = \mu x - x^3 + h \), \( dF/dx = -g'(x) \), so that we can deduce the stability of a fixed point from the sign of the derivative of \( g \). We see that in the case \( \mu < 0 \), the only existing fixed point is stable. When \( \mu > 0 \), the fixed points obtained in the increasing part of \( g \) (the rightmost and leftmost ones) are stable, while the one obtained in the decreasing part (middle fixed point) is unstable.

We can now draw different types of bifurcation diagrams: bifurcation diagrams at fixed \( \mu \) and varying \( h \) and the ones at fixed \( h \) and varying \( \mu \).

The bifurcation diagram for a fixed value of \( \mu \) and considering as a varying parameter \( h \) depends on the sign of \( \mu \) (see figure 2.9).
We see in figure 2.9 that a bifurcation occurs only in the case $\mu > 0$, more precisely the bifurcation diagram displays two saddle-node bifurcations which corresponds in figure 2.7 right to the tangency of the line $y = h$ with the curve $y = g(x)$.

We can also deduce the bifurcation diagram at fixed $h > 0$ and for varying $\mu$ (you can draw the case $h < 0$ by yourself as an exercise):

This bifurcation is called the imperfect pitchfork bifurcation.

Finally the full representation of the stationary solutions in function of the 2 parameters $h$ and $\mu$ is the following one:
2.6 Examples

2.6.1 Start of a laser

*Bibliography: Les lasers, D. Dangoisse, D. Hennequin, V. Zehnlé*

2.6.1.a Model

A laser is a device emitting coherent light. It consists of a gain medium enclosed in a resonant cavity formed by mirrors. At least one of those mirrors is partially transparent, ensuring the output emission.

We will study one of the simplest model for the gain medium: only two states for the atoms constituting this medium will be considered. The states will be called 1 and 2 and the associated energy will be called $E_1$ and $E_2 > E_1$. The lowest level $E_1$ is not the fundamental state. The relationship between the frequency $\nu$ of the light emitted by the
laser and the energy gap $E_2 - E_1$ is:

$$E_2 - E_1 = \hbar \omega = h\nu.$$ 

Let’s defined the following variables:

- $N_1(t)$ is the number of atoms in the state 1 per unit volum at time $t$,
- $N_2(t)$ is the number of atoms in the state 2 per unit volum at time $t$,
- $I(t)$ is the photons flux in the gain medium, i.e. the number of photons crossing a unit surface by unit time.

The mechanisms governing the light emission and absorption have been described by A. Einstein in 1916:

- **absorption**:

  During this process, a photon of energy $h\nu$ is absorbed and an atom switches from state 1 to state 2. The number of atoms shifting from state 1 to state 2 is thus proportional to $N_1$ and $I$. The proportionality coefficient has the dimension of a surface, it is called the *absorption cross section* and will be noted $\sigma_a$. Consequently, we have for the variation of the population of atoms in each states between times $t$ and $t + \Delta t$:

  $$(\Delta N_2)_{\text{abs}} = \sigma_a I N_1 \Delta t$$

  $$(\Delta N_1)_{\text{abs}} = -\sigma_a I N_1 \Delta t$$

To calculate the variation of the photons flux, we consider a cylinder in the gain medium along the direction propagation of the photons:
Per unit time, \( \mathcal{I}(t)S \) photons enter in the volume \( Sc\Delta t \) and \( \mathcal{I}(t+\Delta t)S \) photons exit. Besides, \( \sigma_a \mathcal{I} N_1 \) photons are absorbed per unit time and unit volume, so:

\[
\mathcal{I}(t+\Delta t)S - \mathcal{I}(t)S = -(\sigma_a \mathcal{I} N_1)(c\Delta tS)
\]

\[
(\Delta \mathcal{I})_{abs} = -\sigma_a \mathcal{I} N_1 c\Delta t\]

- **Stimulated emission**: Symetrically to the process of absorption, there is emission of photons stimulated by photons already present in the cavity:

It is very important to note that this emission is not spontaneous: the photons that are generated when the atoms decay from state 2 to state 1 are clones of the photons already present in the gain medium. They share all the properties of the photon that have stimulated the emission, in particular its wave vector. We have for the variations of the quantities that describe our system:

\[
(\Delta N_2)_{sti} = -\sigma_{sti} \mathcal{I} N_2 \Delta t
\]

\[
(\Delta N_1)_{sti} = \sigma_{sti} \mathcal{I} N_2 \Delta t
\]

\[
(\Delta \mathcal{I})_{sti} = \sigma_{sti} \mathcal{I} N_2 c\Delta t
\]

In the case considered here, we have \( \sigma_{sti} = \sigma_a = \sigma \).

- **Losses and decay**: Independantly of the presence of photons already in the cavity, atoms which are not in the fundamental state tend to decay spontaneously towards lower states by emitting photons which are not coherent with the ones constituting the photons flux and consequently do not contribute to \( \mathcal{I} \). Atoms may also decay by other modes than the emission of photons: collisions, vibrations... For the sake
of simplicity, we will consider that the decay rate of the atom population does not depend of the state level:

\[
\begin{align*}
(\Delta N_2)_d &= -\gamma N_2 \Delta t \\
(\Delta N_1)_d &= -\gamma N_1 \Delta t
\end{align*}
\]

Two other effects have to be taken into account. First because of the partially transparent output mirror, a term of loss for the flux has to be included for the evolution of \(I\):

\[
(\Delta I)_{\text{loss}} = -\kappa I \Delta t.
\]

Finally, a laser is an out-of-equilibrium system but we have until now not discussed the energy input in the system. Indeed, with only the terms described right now, the levels would decay until they are empty and the photons flux would decrease until the extinction of the light flux. In a laser, the energy injection consists in a feeding of the higher level, i.e. state 2, enforcing what is called a population inversion. The process is called the “pumping” of the laser and the pumping rate will be denoted \(\lambda\).

Putting everything together, we obtain a model describing the evolution of a laser:

\[
\begin{align*}
\frac{dN_2}{dt} &= \sigma I N_1 - \sigma I N_2 - \gamma N_2 + \lambda \\
\frac{dN_1}{dt} &= -\sigma I N_1 + \sigma I N_2 - \gamma N_1 \\
\frac{dI}{dt} &= -c\sigma I N_1 + c\sigma I N_2 - \kappa I
\end{align*}
\]

We can notice that the relevant variables for the description of the medium are not the individual values of the population of atoms but only the difference \(D = N_2 - N_1\) which is called the population inversion, and the system can be rewritten with only two variables:

\[
\begin{align*}
\dot{D} &= -2\sigma I D - \gamma D + \lambda \\
\dot{I} &= c\sigma I D - \kappa I
\end{align*}
\]

We can non-dimensionnalize the system using the following change of variables: \(D = \frac{\sigma c}{\kappa} D, I = \frac{2\sigma}{\gamma} I, \tau = \gamma t, \) and \(A = \frac{\sigma c}{\kappa \gamma} \lambda\) and we obtain:
where the time variable governing the derivatives is now $\tau$ and the control parameter is $A$, the pumping parameter. Indeed it corresponds to the parameter that can generally easily be tuned experimentally, for example by the variation of an electrical current which is used to excite the atoms to a higher state, or any other mean of pumping. The parameter $k$ depends of the loss of the mirrors through $\kappa$ and of the gain medium through $\gamma$, so that this parameter is constant for a given laser.

2.6.1.b Study of the dynamical system

We thus have to study the dynamical system of dimension 2:

\[
\begin{align*}
\dot{D} &= -D(I + 1) + A \\
\dot{I} &= kI(D - 1)
\end{align*}
\]

Fixed points

\[
\begin{align*}
A &= D(I + 1) \\
0 &= I(D - 1)
\end{align*}
\]

There are two fixed points:

- $I = 0$ and $D = A$, the laser is not emitting any light.
- $D = 1$ and $I = A - 1$, the light flux increases with the pumping while the population inversion is saturated to a constant value.

Jacobian matrix

\[
L = \begin{pmatrix}
-(I + 1) & -D \\
kI & k(D - 1)
\end{pmatrix}
\]

Stability of $(D = A, I = 0)$

\[
L|_{(A,0)} = \begin{pmatrix}
-1 & -A \\
0 & k(A - 1)
\end{pmatrix}
\]

The trace is $T = k(A - 1) - 1$ and the determinant $\Delta = -k(A - 1)$.

- when $A < 1$, $\Delta > 0$ and $T < 0$, the fixed point is stable.
- when $A > 1$, $\Delta < 0$, the fixed point is a saddle point and is unstable.
2.6. EXAMPLES

Stability of \((D = 1, I = A - 1)\)

\[
L|_{(1,A-1)} = \begin{pmatrix} -A & -1 \\ k(A - 1) & 0 \end{pmatrix}
\]

The trace \(T = -A\) is always negative and the determinant is \(\Delta = k(A - 1)\).

- when \(A < 1\), \(\Delta < 0\) the fixed point is a saddle point and is unstable.
- when \(A > 1\), \(\Delta > 0\), the fixed point is stable.

**Bifurcation** From the study of the stability of the fixed points, we deduce that a bifurcation occurs for the value of the control parameter \(A = 1\). There is an exchange of stability between two different fixed points so that it is a *transcritical bifurcation*.

Taking into account the fact that \(I\) and \(A\) are both positive, we can draw the bifurcation diagrams:

The diagrams show that there is a threshold for the start of a laser. The existence of such threshold is typical of a nonlinear system. When the laser is emitting, the population inversion saturates.
2.6.2 Oscillating chemical reactions

2.6.2.a Chemical kinetics

*Bibliography: Instabilités, Chaos et Turbulence, P. Manneville.*

Let’s consider the elementary step of a chemical reaction:

$$\sum_{i=1}^{N} n_i A_i \rightarrow \sum_{i=1}^{N} n'_i A_i$$

where the $A_i$ are the $N$ chemical species implied in the reaction (either reactants or products), of concentration $[A_i]$, $n_i$ (resp. $n'_i$) is the number of mole of $A_i$ before (resp. after) the reaction and can be 0.

The reaction rate is proportional to the number of reactional collisions per unit of time. A reactional collision implies all the reactants at a same location. The probability to have one species $A_i$ at a given location is proportional to its concentration. Consequently, the reaction rate is of the form $k \prod_{i=1}^{n} [A_i]^{n_i}$ with $k$ the reaction rate constant (which depends on the temperature). When a collision happen, $n'_i$ moles of $A_i$ are producted and $n_i$ moles consumed, leading to a variation $n'_i - n_i$ of the number of mole of the species $A_i$. Consequently, the dynamics of the reaction is given by the $N$ equations:

$$\frac{d[A_i]}{dt} = (n'_i - n_i) k \prod_{i=1}^{n} [A_i]^{n_i}$$

If there are several steps to the reaction, the total variation of $[A_i]$ is given by the sum of the variations corresponding to each individual reaction step.

2.6.2.b Belousov-Zhabotinsky oscillating reaction

*Bibliography: L’ordre dans le chaos, P. Bergé, Y. Pomeau, C. Vidal.*

B. Belousov was a Russian chemist who discovered the existence of oscillating chemical reactions. Such reactions have been studied more deeply afterwards by A. Zhabotinsky in the 60s.

Reactants are: sulfuric acid $H_2SO_4$, malonic acid $CH_2(COOH)_2$, sodium bromate $NaBrO_3$ and cerium sulfate $Ce_2(SO_4)_3$. Those reactants, in aqueous solution with certain concentrations, can lead to oscillations in the concentrations of some ions implied in the reaction. Those oscillations can be visualized using a colored redox indicator (e.g. ferroin).

If the reaction is done by simply mixing a given quantity of reactants in a closed container, the oscillations are transitory. To maintain the reaction, you have to supply continuously some reactants, to stir continuously the solution and to have an outlet for

\[1\text{There are several variants of the reaction.}\]
the overflow.

The dynamical variables of the system are the instantaneous values of the concentration of all the species in the reactor. But in that kind of experiment only a few variables are practically measurable. It can be (see *L’ordre dans le chaos*):

- the electrical potential difference between two electrodes immersed in the solution. The relationship between the measured voltage and the concentration in $Br^-$ (in the case of the reaction described previously) is given by the Nernst equation.

- the transmission of light through the tank at a given wavelength. The Beer-Lambert law is then used to describe the absorption of the light. For example, for $\lambda = 340$ nm, the absorption is mainly due to the $Ce^{4+}$ ions.

Another concern is the parameters which can be tuned in such a system. It can be the temperature, which will change the constants of the reactions $k_j$, or the concentration of the species in input, or the flux rates of the pumps that provide the reactants which changes the time spent by the reactants in the tank.

Concerning the modeling of the experiment, the problem is that such reactions are very complicated and can imply a lot of elementary steps which are not known. In fact the complete reaction schematics of some of those reactions is still debated. Among the tips that help for the modeling of such complicated systems is the fact that the characteristic times of the steps of the reaction can be very different so that for example the faster ones can be considered to be instantaneous. Using that kind of approximation it is possible to consider simplier models with a reduce number of differential equations containing only the dynamics of interest for the phenomenon studied.

2.6.2.c The Bruxellator

*Bibliography:*

- *Nonlinear dynamics and chaos, S. Strogatz*,

- *Instabilités, Chaos et Turbulence, P. Manneville*

From a theoretical point of view, because of the complexity of real chemical reactions, an approach consists in the search of minimal models which will lead to an oscillating dynamics. The “Bruxellator” is such a kinetical model proposed by I. Prigogine and R. Lefever in 1968. They considered the global reaction

$$A + B \rightarrow C + D$$
constituted of four steps of elementary reactions and implying two free intermediates species \(X\) and \(Y\). The sequence of elementary reactions considered is:

\[
\begin{align*}
A &\rightarrow X \\
B + X &\rightarrow Y + C \\
2X + Y &\rightarrow 3X \\
X &\rightarrow D
\end{align*}
\]

The control parameters of the system are the concentration of species A and B and we suppose that those concentrations can be maintained at constant values by a continuous supply. All the reaction constant rates are supposed equal to 1. We note \([A] = A\).

The kinetics laws give us the following system of differential equations:

\[
\begin{align*}
\frac{dX}{dt} &= (1 - 0)A + (0 - 1)BX + (3 - 2)X^2Y + (0 - 1)X \\
\frac{dY}{dt} &= (1 - 0)BX + (0 - 1)X^2Y \\
\frac{dC}{dt} &= (1 - 0)BX \\
\frac{dD}{dt} &= (1 - 0)X
\end{align*}
\]

Note that \(C\) and \(D\) are directly given by \(X\) so that there are finally only two pertinent variables for the understanding of the dynamics of the system: \(X\) and \(Y\). Consequently, we want to study the dynamical system:

\[
\begin{align*}
\frac{dX}{dt} &= A - (B + 1)X + X^2Y \\
\frac{dY}{dt} &= BX - X^2Y
\end{align*}
\]

**Fixed points.** They are given by:

\[
\begin{align*}
A &= [(B + 1) - XY]X \\
BX &= X^2Y
\end{align*}
\]

For \(A > 0\), we have necessarily \(X \neq 0\). We deduce that the only fixed point has for coordinates in the phase space:

\[
X = A \text{ and } Y = \frac{B}{A}.
\]
Jacobian matrix.

\[ \mathcal{L} = \begin{pmatrix} -(B + 1) + 2XY & X^2 \\ B - 2XY & -X^2 \end{pmatrix} \]

Stability of the fixed point.

\[ \mathcal{L}|_{(A, \frac{B}{A})} = \begin{pmatrix} B - 1 & A^2 \\ -B & -A^2 \end{pmatrix} \]

The determinant of the jacobian matrix is \( \Delta = A^2 > 0 \), and the trace is \( T = B - 1 - A^2 \). When \( B < 1 + A^2 \), the fixed point is stable, when \( B > 1 + A^2 \), it is unstable. At the bifurcation, the fixed point shifts from a stable spiral point to an unstable spiral point.

Consequently, if the concentration \( A \) is constant, for low concentration of \( B \), the reaction will reach a stationary state: the concentration of \( C \) and \( D \) are constant after a transient. When increasing \( B \), there is a destabilisation of this state through a Hopf bifurcation leading to a limit cycle. The concentrations in \( C \) and \( D \) then display oscillations.

**Poincaré-Bendixson.** We want to apply the Poincaré-Bendixson theorem for given values of \( A \) and \( B \) with \( B > 1 + A^2 \).

We search the nullclines, i.e. the two curves along which the vector field is parallel to one of the axis of the phase plane.

- The first curve is given by \( A - (B + 1)X + X^2Y = 0 \), which leads to:

  \[ Y = f(X) = \frac{(B + 1)X - A}{X^2}. \]

To study this function, we compute its derivative: \( f'(x) = \frac{-(B+1)X^2 + 2A}{X^3} \), which changes of sign when \( X = \frac{2A}{B+1} \):

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>( \frac{2A}{B+1} )</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( f )</td>
<td>↗</td>
<td>↘</td>
<td></td>
</tr>
</tbody>
</table>

- The second function is given by \( BX - X^2Y = 0 \), leading to:

  \[ Y = g(X) = \frac{B}{X}. \]
To draw the curves, we need to know the position of the fixed point (which is the intersection of the two nullclines) compared to the maximum of $f(X)$. When the fixed point is unstable:

$$B > A^2 + 1$$
$$B + 1 > 2$$
$$1 > \frac{2}{B + 1}$$
$$A > \frac{2A}{B + 1}$$

The two curves can then be drawn:

To find a domain on the frontier of which the vector field always points inside the domain, we need to find a line of negative slope smaller than the one of the vector field in the case $\dot{Y} < 0$ and $\dot{X} > 0$.

$$\frac{dY}{dX} = \frac{BX - X^2Y}{A - (B + 1)X + X^2Y} = -1 + \frac{A - X}{A - (B + 1)X + X^2Y}$$

when $X > A$, we thus have $\frac{dY}{dX} < -1$ so that any line of negative slope strictly larger than -1 will do, here we used a $-1/2$ slope:
Integration with Matlab We can also for given values of the parameter draw the nullclines (\( f \) in cyan and \( g \) in green), the vector field (in blue, rescaled) and an example of trajectory (in red, after a transient the trajectory settle on the limit cycle):

Figure 2.12: Integration of the system for \( A = 1 \) and \( B = 3 \) superimposed with the nullclines and the general directions of the vector field (the size of the vectors have been rescaled).
Conclusion

We have presented in this chapter several bifurcations, which correspond to the modification of the number and/or the nature of the attractors of a dynamical system.

We have illustrated some of those bifurcation by few examples coming from different field (start of a laser, chemical reactions).

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